ORBITS, SECTIONS, AND INDUCED TRANSFORMATIONS †

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ABSTRACT

A general me hod of constructing induced transformations in ergodic theory is introduced and studied. We show how this new method generalizes the well known construction of induced transformations defined by a recurrent transformation via the first return time to a subset of positive measure. Using our method we show a connection between ergodic transformations that preserve an equivalent measure and those that preserve no equivalent measure.

1. Introduction

Induced transformations on subsets A of X by a recurrent transformation T defined via the first return time to A under the positive powers of T were first discussed in [7]. In the past thirty years induced transformations have been utilized effectively in constructing numerous examples of ergodic transformations of interest in the subject. Examples of weak mixing transformations, ergodic measure-preserving transformations defined on an infinite measure space, and flows built under a function have been some of the significant outgrowths of the notion of induced transformations. They have also been the subject of study by a number of authors in the field of ergodic theory. There exists an extensive literature in this direction; we cite among these [1], [2], [3], [13], and the references mentioned in these articles where many interesting properties of induced transformations and their connection with entropy or spectral theory are in-

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vestigated. In time it became clear that there was a close connection between induced transformations and the notion of a full group associated with a measurable transformation; the behaviour of the orbit structure of a transformation was also significant in this connection.

Until recently however, the original notion of an induced transformation could not be utilized in the construction of examples of transformations that did not preserve any σ-finite measure equivalent to a given measure. Recently in [5] an attempt in this direction was successful and some simple examples were exhibited. In this article we continue to study induced transformations in more detail. In Section 3 we generalize the notion of induced transformations in a manner that encompasses as a special case the well-established notion of induced transformations that has been studied and utilized until now. Moreover, the general notion of induced transformations by a pair of transformations (T, S) that is introduced in Section 3 together with a number of the examples discussed there prepare the machinery for a more detailed study of the subject subsequently in Section 4. We refer the reader to Section 2 where we list most of the pertinent definitions and notations that are used throughout the paper. We also list in that section a number of lemmas that are needed later. Lemmas 1 through 8 that we have listed in Section 2 have been cited in one form or another in past literature. Some of these lemmas are results in the subject of ergodic theory that may be termed "well known"; others have been stated and proven by various authors at different times. For the sake of completeness we list these results as lemmas and include their proofs. After the formal definition of the generalized notion of induced transformations by a pair (T, S) and a number of illustrative examples in Section 3, we proceed to Section 4. Theorems 1 through 4 of Section 4 are new in nature and describe the behaviour of induced transformations and study relationships that exist among collections of transformations possessing different properties. A few of the corollaries, namely those that follow Theorem 1 and Theorem 4, are results that can be deduced easily from the work of W. Krieger (see [8], for example.) A recent article by D. N. Nghiem [14] came to our attention while the present work was in preparation; part of the results discussed in Theorem 4 seems to be mentioned in [14]. Finally, we would like to call attention of the reader to the work of D. Maharam [11], [12] that seems to be very closely connected in spirit with our work; it would be of interest to study this connection further.

2. Definitions and preliminaries

Let $(X,\mathscr{B},m)$ be a measure space. In what follows, we shall assume that all measures mentioned are non-atomic and σ-finite, and all sets that are considered are measurable either by construction or assumption. For simplicity of notation, expressions involving sets or functions will be stated disregarding sets of measure zero. A measure μ defined on $(X,\mathscr{B})$ is said to be equivalent to m, $\mu \sim m$, in case $\mu(A) = 0$ if and only if $m(A) = 0$. Let $\mathscr{G}(X) = \mathscr{G}(X,\mathscr{B},m)$ be the group of all one-to-one, measurable, and non-singular transformations T of X onto iteself. A set A of positive measure is said to be a weakly wandering set for a transformation $T\in\mathscr{G}(X)$ in case there exists a sequence of integers $\{n_i\}$ such that $\{T^{n_i}A\}$ are mutually disjoint, and A is said to be a wandering set for T in case the sequence $\{n_i\}$ is the set of all integers $\boldsymbol{Z}$. For a transformation $T\in\mathscr{G}(X)$ and a point $x\in X$ we denote by $\mathrm{Orb}_T(x) = \{T^n x \mid n\in\boldsymbol{Z}\}$. A subset A of X is said to be an S-section for a transformation $S\in\mathscr{G}(X)$ if for all $x\in X$, $\mathrm{Orb}_S(x)\cap A$ consists of a single point. Let N be the set of all positive integers; a transformation $S\in\mathscr{G}(X)$ is dissipative if $X = \bigcup_{n\in\boldsymbol{Z}} S^n A$ (disj) where A is a wandering set for S, and S is periodic of period n for $n\in N$ if for some subset A of X we have $X = \bigcup_{0\leq i<n} S^i A$ (disj) and such that $S^n x = x$ for $x\in A$. We denote by $\mathscr{D}(X) = \{S\in\mathscr{G}(X) \mid S \text{ is dissipative}\}$, by $\mathscr{P}_n(X) = \{S\in\mathscr{G}(X) \mid S \text{ is periodic of period } n\}$ for $n\in N$, and by $\mathscr{S}(X) = \{S\in\mathscr{G}(X) \mid S \text{ admits an } S\text{-section}\}$. It is clear that if $S\in\mathscr{D}(X)\cup\bigcup_{n\in N}\mathscr{P}_n(X)$ then $S\in\mathscr{S}(X)$; conversely, we have Lemma 1.

LEMMA 1. *Let $S\in\mathscr{S}(X)$; then there exists a decomposition of $X = X_\infty \cup \bigcup_{n\in N} X_n$(disj) such that*

$$Sx = \begin{cases} S_\infty x \text{ for } x\in X_\infty \\ S_n x \text{ for } x\in X_n,\ n\in N, \end{cases}$$

where $S_n\in\mathscr{P}_n(X_n)$ and $S_\infty\in\mathscr{D}(X_\infty)$.

PROOF. Let $S\in\mathscr{S}(X)$ and let A be an S-section. For $n\in N$ we let $A_n = \{x\in A \mid S^n x\in A, \text{ and } S^i x\notin A \text{ for } 0<i<n\}$ and $A_\infty = \{x\in A \mid S^n x\notin A \text{ for all } n\in N\}$. It is clear then that $X_n = \bigcup_{0\leq i<n} S^i A_n$ (disj) and $X_\infty = \bigcup_{i\in\boldsymbol{Z}} S^i A_\infty$ (disj) gives the desired decomposition and completes the proof.

LEMMA 2. *Let* $\mathscr{H} \subset \mathscr{G}(X)$ *be a group of transformations. Then the following two conditions are equivalent*:

(a) $m(A) > 0$ *and* $TA = A$ *for all* $T \in \mathscr{H}$ *implies* $m(X - A) = 0$.

(b) $m(A) > 0$ *and* $m(B) > 0$ *implies that there exists* $T \in \mathscr{H}$ *such that* $m(TA \cap B) > 0$.

PROOF. It is obvious that (b) implies (a).

Conversely, without loss of generality we may assume that $m(X) < \infty$. Suppose (b) does not hold. Then there exist two subsets A and B of X of positive measure such that $m(TA \cap B) = 0$ for all $T \in \mathscr{H}$. We let $\mathscr{C} = \{C \subset X \mid A \subset C$ and $m(TC \cap B) = 0$ for all $T \in \mathscr{H}\}$, and we let $a = \sup_{C \in \mathscr{C}} m(C)$. Then there exists an increasing sequence of sets $\{C_n \in \mathscr{C},\ n \in N\}$ such that $\lim_{n \to \infty} m(C_n) = a$. We let $A^* = \bigcup_{n \in N} C_n$. It is clear that $A^* \supset A$, $m(TA^* \cap B) = 0$ for all $T \in \mathscr{H}$, and $m(A^*) = a$. In case $T'A^* \neq A^*$ for some $T' \in \mathscr{H}$, there exists $T'' \in \mathscr{H}$ such that $C' = A^* \cup T''A^*$ and $m(C') > a$. Then $C' \supset A$ and $m(TC' \cap B) = 0$ for all $T \in \mathscr{H}$; therefore, $C' \in \mathscr{C}$. This is a contradiction, which implies that $TA^* = A^*$ for all $T \in \mathscr{H}$. But this gives a contradiction to (a) and completes the proof.

For a subset $\mathscr{T}$ of $\mathscr{G}(X)$ we shall say that $\mathscr{T}$ is ergodic if the group generated by $\mathscr{T}$ satisfies either condition (a) or condition (b) of Lemma 2. Since $(X, \mathscr{B}, m)$ is non-atomic it follows that if $T \in \mathscr{G}(X)$ is ergodic, then for any two subsets A and B of X of positive measure there exists a positive integer $n \in N$ such that $m(T^n A \cap B) > 0$. We denote by $\mathscr{E}(X) = \{T \in \mathscr{G}(X) \mid T \text{ is ergodic}\}$. It follows that $\mathscr{E}(X) = \mathscr{E}_I(X) \cup \mathscr{E}_{II}(X) \cup \mathscr{E}_{III}(X)$ (disj), where $\mathscr{E}_I(X) = \{T \in \mathscr{E}(X) \mid T \text{ preserves a measure } \mu \sim m \text{ with } \mu(X) = 1\}$, $\mathscr{E}_{II}(X) = \{T \in \mathscr{E}(X) \mid T \text{ preserves a measure } \mu \sim m \text{ with } \mu(X) = \infty\}$, and $\mathscr{E}_{III}(X) = \{T \in \mathscr{E}(X) \mid T \text{ preserves no measure } \mu \sim m\}$.

For a subset A of X of positive measure we denote by $(A, \mathscr{B}_A, m)$ the induced measure space on A, where $\mathscr{B}_A =$ all measurable subsets of A, and m is the restriction on $(A, \mathscr{B}_A)$ of the measure m given on $(X, \mathscr{B})$. Let $T \in \mathscr{G}(X)$ be a recurrent transformation; that is, for every subset B of X of positive measure and for almost all $x \in B$ we have $T^n x \in B$ for some $n \in N$. We denote by T_A the induced transformation on the subset A of X by T; namely, T_A is defined by the first return time to A under the transformation T; in other words, $T_A x = T^n x$ for $x \in A$ where n is the smallest positive integer such that $T^n x \in A$. We denote the full group of T by $[T] = \{S \in \mathscr{G}(X) \mid Sx = T^n x \text{ for all } x \in X \text{ where } n = n(x) \in Z\}$. We also consider the positive elements of the full group of T; namely,

$[T]^+ = \{S \in \mathscr{G}(X) \mid Sx = T^n x$ for all $x \in X$ where $n = n(x) \in N\}$. For any two subsets A and B of X we say $A \sim B$ by T if there exists a decomposition $A = \bigcup_{n \in N} A_n$(disj) such that $B = \bigcup_{n \in N} T^{k_n} A_n$(disj) for some sequence of integers $\{k_n\}$. In the following Lemma 3 we list some of the basic properties of induced transformations as defined above.

LEMMA 3. *Let $T \in \mathscr{G}(X)$ be a recurrent transformation, and let T_A be the induced transformation by T on a subset A of X of positive measure. Then*

(a) $T_A \in \mathscr{G}(A)$;

(b) *if we let $T_A x = x$ for $x \in X - A$ then $T_A \in [T]$;*

(c) *for two subsets C and D of A if $T_A^n C = D$ for some $n \in N$ then $C \sim D$ by T. If we further assume that $X = \bigcup_{n \in N} T^n A$, then*

(d) *$T \in \mathscr{E}(X)$ if and only if $T_A \in \mathscr{E}(A)$;*

(e) *T preserves a measure $\mu \sim m$ on $(X, \mathscr{B})$ if and only if T_A preserves a measure $\mu \sim m$ on $(A, \mathscr{B}_A)$.*

PROOF. Follows from the definitions.

LEMMA 4. *Let $T \in \mathscr{E}(X)$, and let A and B be two subsets of X of positive measure. Then there exists a decomposition of $A = \bigcup_{i \in N} A_i$(disj) such that for each $i \in N$ we have $T^{n_i} A_i \subset B$ for some integer n_i.*

PROOF. Since T is ergodic we have $A \subset \bigcup_{i \in N} T^i B$. We let $A_1 = A \cap T^{-n_1} B$ where n_1 is the smallest positive integer such that $m(A \cap T^{-n_1} B) > 0$. Next we define A_i for $i > 1$ inductively by $A_i = (A - \bigcup_{1 \leqq k < i} A_k) \cap T^{-n_i} B$ where n_i is the smallest positive integer such that $m[(A - \bigcup_{1 \leqq k < i} A_k) \cap T^{-n_i} B] > 0$, and this completes the proof of the lemma.

By a slight modification of the proof of Lemma 4 we obtain Lemma 5 whose proof we omit. (See also [4, Lem. 6].)

LEMMA 5. *Let $T \in \mathscr{E}(X)$, and let A and B be two subsets of X of positive measure. Then $A \sim B'$ by T for some $B' \subset B$, or $B \sim A'$ by T for some $A' \subset A$.*

COROLLARY. *Let $T \in \mathscr{E}(X)$, and let $\mu \sim m$ on $(X, \mathscr{B})$ be an invariant measure for T. Then for any two sets A and B with $\mu(A) = \mu(B)$ we have $A \sim B$ by T.*

PROOF. In case $\mu(A) = \mu(B) < \infty$ the proof follows from Lemma 5 and the fact that μ is an invariant measure for T. In case $\mu(A) = \mu(B) = \infty$, then since μ is σ-finite, we decompose $A = \bigcup_{n \in N} A_n$(disj) and $B = \bigcup_{n \in N} B_n$ (disj) such that $\mu(A_n) = \mu(B_n) = 1$ for $n \in N$. It follows that $A_n \sim B_n$ for $n \in N$, and this completes the proof.

LEMMA 6. *Let $T \in \mathscr{E}_{III}(X)$; then for any set $A \subset X$ of positive measure we have $A \sim X$ by T. Hence $A \sim B$ by T for any two sets A and B of positive measure.*

PROOF. We assume that $B = X - A$ has positive measure. Let T_A be the induced transformation by T on the set A. From Lemma 3 part (e) follows that T_A does not preserve any measure $\mu \sim m$ on $(A, \mathscr{B}_A)$. From [6, Th. 1] follows that T_A admits a weakly wandering set $W \subset A$. In other words, there exists a set $W \subset A$ such that $m(W) > 0$ and

(1) $$A \supset \bigcup_{i \in N} T_A^{p_i} W \text{ (disj)}.$$

Next we let T_W be the induced transformation by T on the set W. Again it follows that there exists a set $V \subset W$ such that $m(V) > 0$ and

(2) $$W \supset \bigcup_{i \in N} T_W^{q_i} V \text{ (disj)}.$$

From Lemma 4 follows that

(3) $$B = \bigcup_{i \in N} B_i \text{(disj) and } T^{n_i} B_i \subset V \text{ for } i \in N.$$

From (2), (3), and part (c) of Lemma 3 follows that there exists a set $E \subset W$ such that $B \sim E$; where $E = \bigcup_{i \in N} T_W^{q_i}(T^{n_i} B_i)$ (disj). We also have from (1) that $E \sim E_i$ for $i \in N$, where $E_i = T_A^{p_i} E \subset T_A^{p_i} W$. It follows that if $F = E \cup \bigcup_{i \in N} E_i$(disj), then $F \sim F - E$ by T. Finally, we have

$$X = B \cup F \cup (A - F) \sim E \cup (F - E) \cup (A - F) = A$$

by T, and this completes the proof.

COROLLARY. *Let $T \in \mathscr{E}_{III}(X)$, and let C be a subset of X of positive measure. Suppose $C = \bigcup_{0 \leqq i < n} A_i$(disj) for some $n \in N$, where $m(A_i) > 0$ for $0 \leqq i < n$. Then there exists a transformation $S \in [T]$ with $S \in P_n(C)$ such that $SA_i = A_{i+1 (\text{mod } n)}$ and $Sx = x$ for $x \in X - C$.*

PROOF. For each i with $0 \leqq i < n$ it follows from Lemma 6 that $A_i \sim A_{i+1}$ by T. This defines a transformation $S: A_i \to A_{i+1}$ where $Sx = T^k x$, $k = k(x) \in \mathbf{Z}$, and $x \in A_i$. If we let $Sx = S^{-n}x$ for $x \in A_{n-1}$ and $Sx = x$ for $x \in X - C$, it follows that $S \in [T]$ and $S \in \mathscr{P}_n(C)$; and this completes the proof.

For a transformation $T \in \mathscr{G}(X)$ and a measure $\mu \sim m$ on $(X, \mathscr{B})$ we denote by μT the measure defined on $(X, \mathscr{B})$ by $(\mu T)(A) = \mu(TA)$ for all $A \in \mathscr{B}$. For two measures μ and ν with $\mu \sim \nu$ on $(X, \mathscr{B})$ and for $x \in X$ we denote by $(d\nu/d\mu)(x)$ the Radon-Nikodym derivative of ν with respect to μ evaluated at x. Let $\boldsymbol{R}$ be the set of all real numbers; we shall say that the measure ν is piecewise linear with respect to the measure μ in case $X = \bigcup_{\alpha \in \Lambda} A_\alpha(\text{disj})$ where $A_\alpha = \{x \in X \mid (d\nu/d\mu)(x) = \alpha\}$ for all $\alpha \in \boldsymbol{R}$ and $\Lambda = \{\alpha \in \boldsymbol{R} \mid \mu(A_\alpha) > 0\}$.

LEMMA 7. *Let $T \in \mathscr{G}(X)$ and let $\mathscr{H}(\mu) = \{Q \in [T] \mid \mu Q = \mu\}$ be an ergodic set for some measure $\mu \sim m$ on $(X, \mathscr{B})$. Then there exists an ergodic transformation $V \in [T]^+$ such that $[V] = \mathscr{H}(\mu)$.*

PROOF. Let $B = \{x \in X \mid (d(\mu T^n)/d\mu)(x) \neq 1 \text{ for all } n \in N\}$. For $x \in B$ and $Q \in \mathscr{H}(\mu)$ with $Qx \neq x$ we have $Qx = T_x^k$ where $k = k(x) < 0$, and $x = Q^{-1}(Qx) = T^j(Qx)$ where $j = -k = -k(x) > 0$; which implies that $Qx \notin B$. In case $m(B) > 0$, let C and D be two subsets of B of positive measure. It follows that $m(QC \cap D) = 0$ for any $Q \in \mathscr{H}(\mu)$. This contradicts the ergodicity of $\mathscr{H}(\mu)$ and shows that $m(B) = 0$. Therefore, for almost all $x \in X$ we define $Vx = T^n x$ where $n = n(x) > 0$ is the smallest positive integer such that $(d(\mu T^n)/d\mu)(x) = 1$. It follows that $V \in [T]^+$ and $[V] = \mathscr{H}(\mu)$, and this completes the proof.

For a transformation $T \in \mathscr{G}(x)$ and a measure $\mu \sim m$ on $(X, \mathscr{B})$ suppose that the measure μT is piecewise linear with respect to μ. We shall denote by $\Delta(\mu)$ the multiplicative group generated by Λ where $\Lambda = \{\alpha \in \boldsymbol{R} \mid \mu(A_\alpha) > 0\}$ and $A_\alpha = \{x \in X \mid (d(\mu T)/d\mu)(x) = \alpha\}$ for $\alpha \in \boldsymbol{R}$.

LEMMA 8. *Let $T \in \mathscr{E}_{III}(X)$ and let $V \in [T]^+$ such that V is ergodic and preserves a measure $\mu \sim m$ on $(X, \mathscr{B})$. If μT is piecewise linear with respect to μ then for almost all $x \in X$ we have $\Delta = \{(d(\mu T^n)/d\mu)(x) \mid n \in N\}$ where $\Delta = \Delta(\mu)$ is the multiplicative group described above.*

PROOF. We consider the set of real numbers Λ and the sets A_α for $\alpha \in \boldsymbol{R}$ as described above. We denote by $\tilde{\Lambda}$ the set of all real numbers $\beta \in \boldsymbol{R}$ such that $\mu\{x \in X \mid (d(\mu T^n)/d\mu)(x) = \beta\} > 0$ for some positive integer $n \in N$. For $\beta \in \tilde{\Lambda}$ let $n = n(\beta) \in N$ be the smallest positive integer such that the set

$$A_{\beta,n} = \{x \in X \mid (d(\mu T^n)/d\mu)(x) = \beta\} \tag{4}$$

has positive measure. For $\beta \in \tilde{\Lambda}$ and $x \in A_{\beta,n}$ we have

$$(5) \qquad \beta = \frac{d(\mu T^n)}{d\mu}(x) = \frac{d(\mu T)}{d\mu}(T^{n-1}x)\cdot\frac{d(\mu T)}{d\mu}(T^{n-2}x)\cdot\dots\cdot\frac{d(\mu T)}{d\mu}(x)$$
$$= \alpha_1\alpha_2\cdots\alpha_n,$$

where $\alpha_i = (d(\mu T)/d\mu)(T^{n-i}x)\in\Lambda$ for $1 \leqq i \leqq n$. We conclude from (4) and (5) that $\Delta \supset \tilde{\Lambda} \supset \Lambda$. Next we let $\beta,\gamma\in\tilde{\Lambda}$; then there exist integers $i,j\in N$ such that $\mu(A_{\beta,i}) > 0$ and $\mu(A_{\gamma,j}) > 0$. Since $V\in[T]^+$ is an ergodic transformation, there exists a positive integer $k > i-j$ such that $\mu(V^{-k}A_{\beta,i}\cap A_{\gamma,j}) > 0$; there also exists a positive integer $p \geqq k$ and a subset $B_{\beta,i}$ of $A_{\beta,i}$ such that $\mu(V^{-k}B_{\beta,i}\cap A_{\gamma,j}) > 0$ and $V^kx = T^px$ for all $x\in V^{-k}B_{\beta,i}$. We let $C = T^j(V^{-k}B_{\beta,i}\cap A_{\gamma,j})$; then $\mu(C) > 0$, and for $x\in C$ we have

$$(6) \qquad \frac{d(\mu T^{-j})}{d\mu}(x) = \gamma^{-1} \text{ since } x\in T^jA_{\gamma,j},$$

$$(7) \qquad \frac{d(\mu V^k)}{d\mu}(T^{-j}x) = 1 \text{ since } \mu V = \mu,$$

$$(8) \qquad \frac{d(\mu T^i)}{d\mu}(V^kT^{-j}x) = \beta \text{ since } V^kT^{-j}x\in B_{\beta,i}, \text{ and}$$

$$(9) \qquad V^k(T^{-j}x) = T^p(T^{-j}x) \text{ since } T^{-j}\in V^{-k}\ x\ B_{\beta,i}.$$

Therefore, for $x\in C$ we have from (6), (7), (8), and (9)

$$\frac{d(\mu T^{i+p-j})}{d\mu}(x) = \frac{d(\mu T^i)}{d\mu}(V^kT^{-j}x)\cdot\frac{d(\mu V^k)}{d\mu}(T^{-j}x)\cdot\frac{d(\mu T^{-j})}{d\mu}(x) = \beta\gamma^{-1},$$

and this implies that $\tilde{\Lambda}$ is a group which proves that $\tilde{\Lambda} = \Delta$.

Next we let $N = X - \bigcup_{\delta\in\Delta}\bigcup_{q\in N}V^{-q}A_{\delta,k}$. Since Δ is countable and V is ergodic it follows that $\mu(N) = 0$. Let $x\in X - T^{-1}N$ and $\delta\in\Delta$; then $x\in A_\alpha$ for some $\alpha\in\Lambda$. We let $\delta' = \delta\alpha^{-1}\in\Delta$. It follows that $V^qTx\in A_{\delta',k'}$ for some $q > 0$. Then

$$\frac{d(\mu T^{k'}V^qT)}{d\mu}(x) = \frac{d(\mu T^{k'})}{d\mu}(V^qTx)\cdot\frac{d(\mu V^q)}{d\mu}(Tx)\cdot\frac{d(\mu T)}{d\mu}(x) = \delta'\alpha = \delta;$$

and this completes the proof of the lemma.

3. A general definition of induced transformations

LEMMA 9. *Let $S\in\mathscr{S}(X)$ and let A be an S-section. A subset B of X is an S-section if and only if there exists an $S'\in[S]$ such that $S'A = B$; moreover, if for some $S''\in[S]$ $S''A = B$, then $S'x = S''x$ for all $x\in A$.*

PROOF. Follows from the definitions.

For $T \in \mathscr{G}(X)$ and $S \in \mathscr{S}(X)$ let A be an S-section such that $T^{-1}A$ is again an S-section. Let $(A, \mathscr{B}_A, m)$ be the induced measure space on A, then using Lemma 9 we define for $x \in A$,

(11) $\quad Rx = TS'x$ where $S' \in [S]$ and $S' : A \to T^{-1}A$.

It is clear that such an R is uniquely defined on the measure space $(A, \mathscr{B}_A, m)$; in fact, $R \in \mathscr{G}(A)$.

DEFINITION 1. For $T \in \mathscr{G}(X)$ and $S \in \mathscr{S}(X)$ let A be an S-section such that $T^{-1}A$ is again an S-section. The transformation $R \in \mathscr{G}(A)$ defined by (11) is said to be the transformation induced by the pair (T, S) on the S-section A.

In the special cases, as given in Example 1, the above definition gives rise to what is generally referred to in the literature as the induced transformation T_A as defined in the preceding section.

EXAMPLE 1. Let $T \in \mathscr{G}(X)$ be a recurrent transformation, and let A be a subset of X of positive measure. For $n \in N$ we let $A_n = \{x \in A \mid T^n x \in A$ and $T^i x \notin A$ for $0 < i < n\}$, and let $X_n = \bigcup_{0 \leqq i < n} T^i A_n$(disj). It is clear that $X = \bigcup_{n \in N} X_n$ (disj). Next we define

$$Sx = \begin{cases} Tx \text{ if } x \in T^i A_n \text{ for } 0 \leqq i < n-1,\ n \in N \\ T^{-(n-1)}x \text{ if } x \in T^{n-1}A_n \text{ for } n \in N. \end{cases}$$

It follows that $S \in [T]$, $S \in \mathscr{S}(X)$, and A is an S-section such that $T^{-1}A$ is again an S-section. Therefore, we can define the transformation T_A induced by the pair (T, S) on the S-section A; namely, $T_A x = TS'x$ for $x \in A$, where $S' \in [S]$ and $S' : A \to T^{-1}A$.

Other properties of the transformations T_A constructed as in Example 1 have been investigated extensively in the literature; see [1], [2], [3], [7], and [13]. Lemma 10 enables us to obtain a whole new collection of examples somewhat different from the ones mentioned up to now.

LEMMA 10. *Let* $T \in \mathscr{G}(X)$ *and* $S \in \mathscr{D}(X)$, *then the following conditions are equivalent*:

(a) $T[S] = [S]T$;

(b) T *maps* $\mathrm{Orb}_S(x)$ *onto* $\mathrm{Orb}_S(Tx)$ *for every* $x \in X$;

(c) A *is an* S*-section if and only if* $T^{-1}A$ *is an* S*-section.*

PROOF. Follows from the definitions.

EXAMPLE 2. Let $T \in \mathscr{G}(X)$, $S \in \mathscr{D}(X)$, and suppose $T[S] = [S]T$. Therefore, according to Lemma 10, for any S-section A we can define the transformation $R \in \mathscr{G}(A)$ induced bt the pair (T, S) on A. Namely, $Rx = TS'x$ for $x \in A$, where $S' \in [S]$ and $S': A \to T^{-1}A$.

The study of such transformations was started in [5] where a concrete example was presented. In the following discussion we shall study them further. At this point however, we introduce another related construction which in a sense is the converse of the notion of an induced transformation $R \in \mathscr{G}(A)$ by a pair (T, S) on an S-section A. The following construction, in the special cases, corresponds to what is generally referred to in the literature as building up a transformation by the stacking or skyscraper construction. (See [3], [7].)

DEFINITION 2. Let $(A, \mathscr{B}_A, m)$ be a measure space, and let $R \in \mathscr{G}(A)$. We shall say that a pair (T, S) of transformations defined on a measure space $(X, \mathscr{B}, m)$ is a pair built up on the transformation $R \in \mathscr{G}(A)$ in case $(X, \mathscr{B}, m)$ induces the measure space $(A, \mathscr{B}_A, m)$ on the subset A of X, $T \in \mathscr{G}(X)$, $S \in \mathscr{S}(X)$, A is an S-section while $T^{-1}A$ is again an S-section, and the transformation induced by the pair (T, S) on the S-section A, according to Definition 1, is the given transformation $R \in \mathscr{G}(A)$.

Given a measure space $(A, \mathscr{B}_A, m)$ and a transformation $R \in \mathscr{G}(A)$ we shall construct in the sequel pairs (T, S) of transformations on a measure space $(X, \mathscr{B}, m)$ built up on the transformation $R \in \mathscr{G}(A)$. Such a construction is not unique. The different methods of constructing them however, produce a number of interesting examples. Moreover, such a construction enables us to study properties of a given transformation $R \in \mathscr{G}(A)$ in a different setting. In some cases, as seen in the following section, we are able to describe properties of a given transformation $R \in \mathscr{G}(A)$ by a "somewhat well-behaved" pair of transformations (T, S) built up on the transformation $R \in \mathscr{G}(A)$.

EXAMPLE 3. Let $(A, \mathscr{B}_A, m)$ be a measure space, and let $R \in \mathscr{G}(A)$. Let $A = \bigcup_{n \in N} A_n$ (disj) be a decomposition of A. For $n \in N$ we let $X_n = \bigcup_{0 \leqq i < n} (A_n, i)$(disj) where $(A_n, 0) = A_n$ and (A_n, i) is an isomorphic copy of A_n for $0 \leqq i < n$. For each $n \in N$, $0 \leqq i < n$, and $x = (y, i) \in X_n$ we let $P_n(y, i) = (y, i + 1 (\mathrm{mod}\, n))$ where $y \in A_n$. Next we form the measure space $(X, \mathscr{B}, m)$ where $X = \bigcup_{n \in N} X_n$(disj); $\mathscr{B}$ = σ-field generated by all sets of the form $(B, i) \subset (A_n, i)$

for $B \subset A_n$, $0 \leqq i < n$, $n \in N$; and m is the extension to $(X, \mathscr{B})$ of the measure m defined on $(A, \mathscr{B}_A)$; namely, $m[(B, i)] = m(B)$ for $(B, i) \subset (A_n, i)$, $B \subset A_n$, $0 \leqq i < n$, and $n \in N$. For $x \in X_n$, $n \in N$, we let $Sx = P_n x$. We also extend R to X by letting $Rx = x$ for $x \in X - A$. Finally, we define T on X by $Tx = RSx$ for $x \in X$. It is clear that $S \in \mathscr{S}(X)$, A is an S-section while $T^{-1}A$ is again an S-section; and the transformation induced on the S-section A by the pair (T, S) is the given transformation $R \in \mathscr{G}(A)$. Therefore, the pair (T, S) of transformations defined on $(X, \mathscr{B}, m)$ is a pair built up on the transformation $R \in \mathscr{G}(A)$.

Transformations T obtained as in Example 3 on the measure space $(X, \mathscr{B}, m)$ are the ones usually referred to in the literature as those built up from the transformation R on top of the set A via the skyscraper or stacking method. Ordinarily, the transformation S as described in Example 3 is suppressed in the literature when similar constructions are discussed.

In Example 4 we construct, by our method, pairs (T, S) of transformations on a measure space $(X, \mathscr{B}, m)$ built up on a transformation $R \in \mathscr{G}(A)$ which are different from the ones constructed by the skyscraper or stacking method. In what follows we shall study in more detail the behaviour of transformations constructed as in Example 4.

EXAMPLE 4. Let $(A, \mathscr{B}_A, m)$ be a measure space and let $R \in \mathscr{G}(A)$. We consider the measure space $(X, \mathscr{B}, m)$ where $X = \bigcup_{n \in \mathbf{Z}} (A, n)$, $\mathscr{B}$ is the σ-field generated by all sets of the form (B, n) for $B \subset A$ and $n \in \mathbf{Z}$, and m is the extension to $(X, \mathscr{B})$ of the measure m defined on $(A, \mathscr{B}_A)$; namely, $m[(B, n)] = m(B)$ for $B \subset A$ and $n \in \mathbf{Z}$. We identify $(A, 0)$ with A. For $x = (y, n) \in X$ we define $Sx = (y, n+1)$ and $\tilde{R}x = (Ry, n)$ where $y \in A$ and $n \in \mathbf{Z}$. Finally, we let $\mathscr{T} = \tilde{R}[S]$. It is clear that $S \in \mathscr{D}(X)$, A is an S-section, $T[S] = [S]T$ for any $T \in \mathscr{T}$, and the transformation induced by the pair (T, S) for any $T \in \mathscr{T}$ on the S-section A is equal to the given transformation $R \in \mathscr{G}(A)$. In other words, for any $T \in \mathscr{T}$ the pair of transformations (T, S) defined on the measure space $(X, \mathscr{B}, m)$ is a pair built up on the transformation $R \in \mathscr{G}(A)$.

4. Main results

THEOREM 1. *Let $(X, \mathscr{B}, m)$ be a measure space and let $T \in \mathscr{E}_{III}(X)$. Let $R \in \mathscr{E}_I(A)$ where $A \subset X$ with $0 < m(A) < m(X)$, and let $Rx = x$ for $x \in X - A$. Then there exist two transformations S_1, $S_2 \in [T]$ such that $RS_1 = T_1 \in \mathscr{E}_I(X)$ and $RS_2 = T_2 \in \mathscr{E}_{II}(X)$. Furthermore, for each $i = 1, 2$, A is an S_i-section while*

$T_i^{-1}A$ is again an S_i-section; and the transformation induced by the pair (T_i, S_i) on the S_i-section A is the given transformation $R \in \mathscr{E}_I(A)$.

PROOF. Let $\mu \sim m$ be the invariant measure for $R \in \mathscr{E}_I(A)$ defined on $(A, \mathscr{B}_A)$ with $\mu(A) = 1$, and let $B = X - A$. Since $T \in \mathscr{E}_{III}(X)$, $m(A) > 0$, and $m(B) > 0$, by the corollary to Lemma 6 there exists a transformation $S_1 \in [T]$ such that $S_1 \in \mathscr{P}_2(X)$ and $S_1 A = B$. For $C \in \mathscr{B}$ we let $\mu(C) = \mu(C \cap A) + \mu[S_1^{-1}(C \cap B)]$. It follows that $RS_1 = T_1 \in \mathscr{E}_I(X)$ with $\mu \sim m$ on $(X, \mathscr{B})$ as the finite invariant measure for T_1.

Next we decompose $A = \bigcup_{n \in N} A_n$(disj) with $\mu(A_n) > 0$ for $n \in N$ and $\sum_{1 \leq n < \infty} n\mu(A_n) = \infty$. We decompose $B = \bigcup_{2 \leq n < \infty} \bigcup_{1 \leq i < n} B_{n,i}$ (disj) such that $\mu(B_{n,i}) > 0$ for $0 < i < n$ and $n > 1$. We let $C_1 = A_1$ and $C_n = A_n \cup \bigcup_{1 \leq i < n} B_{n,i}$ (disj) for $n > 1$. Since $T \in \mathscr{E}_{III}(X)$, by the corollary to Lemma 6 there exists a transformation $S_2 \in [T]$ such that $S_2 \in \mathscr{P}_n(C_n)$ for $n \in N$; and such that $S_2 A_1 = A_1$, for $n > 1$ we have $S_2 A_n = B_{n,1}$, $S_2 B_{n,i} = B_{n,i+1}$ for $0 < i < n-1$, and $S_2 B_{n,n-1} = A_n$. For $C \in \mathscr{B}$ we let $\mu_2(C) = \mu(C \cap A) + \sum_{2 \leq n < \infty} \sum_{1 \leq i < n} \mu[S_2^{-i}(C \cap B_{n,i})]$. It follows that $RS_2 = T_2 \in \mathscr{E}_{II}(X)$ with $\mu_2 \sim m$ on $(X, \mathscr{B})$ as the infinite invariant measure for T_2. The remaining part of the theorem is then clear from the construction, and this completes the proof.

COROLLARY. *Let $T \in \mathscr{E}_{III}(X)$ and let $\mu \sim m$ on $(X, \mathscr{B})$ be an invariant measure for an ergodic transformation $V \in [T]$. Then there exist two transformations $T_1, T_2 \in [T]$ such that $T_1 \in \mathscr{E}_I(X)$ and $T_2 \in \mathscr{E}_{II}(X)$.*

PROOF. Let $\mu \sim m$ on $(X, \mathscr{B})$ be an invariant measure for an ergodic transformation $V \in [T]$. Let A be a subset of X with $0 < \mu(A) < \infty$, and let V_A be the induced transformation by V on the subset A. We extend V_A to X by letting $V_A x = x$ for $x \in X - A$. Then $V_A \in [T]$ and $V_A \in \mathscr{E}_I(A)$. By Theorem 1 there exist two transformations $S_1, S_2 \in [T]$ such that $T_1 = V_A S_1 \in \mathscr{E}_I(X)$ and $T_2 = V_A S_2 \in \mathscr{E}_{II}(X)$. It is clear that $T_1, T_2 \in [T]$, and this completes the proof.

THEOREM 2. *Let $T \in \mathscr{G}(X)$ and $S \in \mathscr{D}(X)$ such that $T[S] = [S]T$. For an S-section $A \subset X$ let R be the transformation induced by the pair (T, S) on A; namely, $Rx = TS'x$ for $x \in A$ where $S' \in [S]$ and $S' \colon A \to T^{-1}A$. We let $\mathscr{T} = T[S]$, then*

(a) *$\mathscr{T} = \{T' \in \mathscr{G}(X) \mid T'[S] = [S]T'$, and the transformation induced by the pair (T', S) on the S-section A equals the transformation $R \in \mathscr{G}(A)\}$;*

(b) *R is ergodic on $(A, \mathscr{B}_A, m)$ if and only if $\mathscr{T}$ is ergodic on $(X, \mathscr{B}, m)$;*

(c) *there exists a measure $\mu \sim m$ on $(A, \mathscr{B}_A)$ and invariant for R if and only if there exists a measure $\mu \sim m$ on $(X, \mathscr{B})$ and invariant for $\mathscr{T}$.*

PROOF. Part (a) follows from the definitions. Suppose R is ergodic on $(A, \mathscr{B}_A, m)$, and let C and D be two subsets of X of positive measure. Then there exist $S_1, S_2 \in [S]$ such that the sets $C' = S_1 C \cap A$ and $D' = S_2 D \cap A$ have positive measure. It follows that $R^n C' \cap D'$ has positive measure for some positive integer n. From the definition of R and the properties of S and T we have

$$R^n C' \cap D' \subset T^n S_n S_1 C \cap S_2 D = S_2(S_2^{-1} T^n S_n S_1 C \cap D) = S_2(T^n S_n' C \cap D)$$

for some $S_n, S_n' \in [S]$. This shows that $T^n S_n' C \cap D$ has positive measure and proves that $\mathscr{T}$ is ergodic on $(X, \mathscr{B}, m)$. Conversely, suppose that $\mathscr{T}$ is ergodic on $(X, \mathscr{B}, m)$ and let C and D be two subsets of A of positive measure. It follows that there exists a transformation V in the group generated by $\mathscr{T}$ such that $VC \cap D$ has positive measure. This implies that $R^n C \cap D$ has positive measure for some integer n and completes the proof of part (b).

To show part (c) we let the measure $\mu \sim m$ on $(X, \mathscr{B})$ be invariant for $\mathscr{T}$. Let $x \in X$, then $x = (y, n)$ where $y \in A$ and $n \in \mathbf{Z}$. Since $\tilde{R}$ defined by $\tilde{R}(y, n) = (Ry, n)$ belongs to $\mathscr{T}$ it follows that μ is an invariant measure for $\tilde{R}$. This implies that the restriction of μ on $(A, \mathscr{B}_A)$ is equivalent to m and is invariant for $R \in \mathscr{G}(A)$. Conversely, let $\mu \sim m$ on $(A, \mathscr{B}_A)$ be an invariant measure for R. We extend μ to $(X, \mathscr{B})$, denote it by the same letter μ, and define it as follows: let $B \in \mathscr{B}$, then $B = \bigcup_{n \in \mathbf{Z}} B_n$(disj) where $B_n = B \cap S^{-n} A$ for $n \in \mathbf{Z}$. We let $\mu(B) = \sum_{n \in \mathbf{Z}} \mu(S^n B_n)$. It is clear that $\mu \sim m$ and is invariant for both S and $\tilde{R}$ on $(X, \mathscr{B})$. If $T' \in \mathscr{T}$ then $T' = \tilde{R} S'$ for some $S' \in [S]$. It follows that μ is also invariant for T' on $(X, \mathscr{B})$, and this completes the proof of the theorem.

REMARK 1. It is clear from the proof of part (c) of Theorem 2 that in case $\mu \sim m$ on $(A, \mathscr{B}_A)$ is an invariant measure for R, then the extended measure $\mu \sim m$ on $(X, \mathscr{B})$ is an invariant measure for any $T' \in \mathscr{T}$ and also for any $S' \in [S]$; in particular, for every S-section $C \subset X$ we have $\mu(C) = \mu(A)$.

LEMMA 11. *Let $(A, \mathscr{B}_A, m)$ be a measure space, and let $R \in \mathscr{G}(A)$ be a transformation for which there does not exist a finite invariant measure $\nu \sim m$ on $(A, \mathscr{B}_A)$. Let the measure space $(X, \mathscr{B}, m)$, the transformation $S \in \mathscr{D}(X)$, and the collection of transformations $\mathscr{T}$ be constructed as in Example 4. Let $\mu \sim m$ on $(X, \mathscr{B})$ be an invariant measure for some $T \in \mathscr{T}$, then there exists an S-section $C \subset X$ with $\mu(C) = \infty$.*

PROOF. Since R does not preserve a finite invariant measure $\nu \sim m$ it follows from [6, Th. 1] that R admits a weakly wandering set $W \subset A$. In other words, there exists a set $W \subset A$ with $m(W) > 0$ and $A \supset \bigcup_{i \in N} R^{n_i} W$ (disj) for some sequence of integers $\{n_i \mid i \in N\}$. Let $\mu \sim m$ on $(X, \mathscr{B})$ be an invariant measure for some $T \in \mathscr{T}$. We let $B = A - \bigcup_{i \in N} R^{n_i} W$; then it is clear that $C = B \cup \bigcup_{i \in N} T^{n_i} W$ is an S-section and $\mu(C) \geqq \sum_{1 \leqq i < \infty} \mu(T^{n_i} W) = \infty$. This completes the proof of the lemma.

THEOREM 3. *Let $(A, \mathscr{B}_A, m)$ be a measure space, and let $R \in \mathscr{E}(A)$. Let the measure space $(X, \mathscr{B}, m)$, the transformation $S \in \mathscr{D}(X)$, and the collection of transformations $\mathscr{T}$ be constructed as in Example 4. Let $\mathscr{M}$ = the set of all measures $\mu \sim m$ on $(X, \mathscr{B})$ such that μ is an invariant measure for some $T \in \mathscr{T}$. Then*

(a) *$R \in \mathscr{E}_I(A)$ implies for any $\mu \in \mathscr{M}$ there exists an S-section $C \subset X$ with $\mu(C) < \infty$;*

(b) *$R \in \mathscr{E}_{II}(A)$ implies for any $\mu \in \mathscr{M}$ there exists an S-section $C \subset X$ with $\mu(C) = \infty$;*

(c) *$R \in \mathscr{E}_{III}(A)$ implies for any $\mu \in \mathscr{M}$, for any real number $c > 0$, and for $c = \infty$ there exists an S-section $C \subset X$ with $\mu(C) = c$.*

PROOF. Let $R \in \mathscr{E}_I(A)$, and let $\mu \sim m$ on $(X, \mathscr{B})$ be an invariant measure for some $T \in \mathscr{T}$. Let $\mu' \sim m$ on $(A, \mathscr{B}_A)$ be the invariant measure for R with $\mu'(A) = 1$, and let B be a subset of A with $0 < \mu(B) < \infty$. Then there exist a positive integer n and a subset B^1 of B with $\mu'(B^1) = 1/n$. It follows that there exist subsets $B^2, B^3, \cdots, B^n$ of A such that $A = \bigcup_{1 \leqq i \leqq n} B^i$(disj) and $\mu'(B^i) = 1/n$ for $1 < i \leqq n$. From the corollary to Theorem 5 follows that $B^1 \sim B^i$ by R for $1 < i \leqq n$. In other words, for each $i = 1, 2, \cdots, n$ there exist a decomposition of $B^1 = \bigcup_{j \in \mathscr{N}} B_{i,j}$ (disj) and a sequence of integers $\{n_{i,j} \mid j \in N\}$ such that $B^i = \bigcup_{j \in N} R^{n_{i,j}} B_{i,j}$ (disj) for $1 \leqq i \leqq n$, where $n_{1,j} = 0$ for $j \in N$. It is clear that $C = \bigcup_{1 \leqq i \leqq n} \bigcup_{j \in N} T^{n_{i,j}} B_{i,j}$(disj) is an S-section. From the fact that μ is an invariant measure for T we have $\mu(C) = n\mu(B^1) \leqq n\mu(B) < \infty$, and this proves part (a).

Part (b) follows from Lemma 11.

To prove part (c) we let $\mu \sim m$ on $(X, \mathscr{B})$ be an invariant measure for some $T \in \mathscr{T}$. From Lemma 11 follows that there exists an S-section A' with $\mu(A') = \infty$. Finally, we let $c > 0$ be a positive number. We let B be a subset of A' with $\mu(B) = c/2$, and let $D = A' - B$. Since A' and A are both S-sections, by Lemma 9 there exists a transformation $S_1 \in [S]$ such that $S_1 A = A'$. It follows that

$R_1 = S_1 R S_1^{-1} \in \mathscr{E}_{III}(A')$, and since $\mu(B) > 0$ and $\mu(D) > 0$ it follows from Lemma 6 that $B \sim D$ by R_1; in other words, there exist a decomposition of $B = \bigcup_{i \in N} B_i$(disj) and a sequence of integers $\{n_i \mid i \in N\}$ such that $D = \bigcup_{i \in N} R_1^{n_i} B_i$(disj). It is clear that $C = B \cup \bigcup_{i \in N} T^{n_i} B_i$(disj) is an S-section, and from the fact that μ is an invariant measure for T it follows that $\mu(C) = 2\mu(B) = c$. This proves part (c) and completes the proof of the theorem.

COROLLARY. *Under the same hypotheses as in Theorem 3, we have*

(a) *$R \in \mathscr{E}_I(A)$ if and only if there exists a measure $\mu \in \mathscr{M}$ such that every S-section $C \subset X$ has finite μ measure;*

(b) *$R \in \mathscr{E}_{II}(A)$ if and only if there exists a measure $\mu \in \mathscr{M}$ such that every S-section $C \subset X$ has infinite μ measure;*

(c) *$R \in \mathscr{E}_{III}(A)$ if and only if for every measure $\mu \in \mathscr{M}$, for every positive number $c > 0$, and for $c = \infty$ there exists an S-section $C \subset X$ with $\mu(C) = c$.*

PROOF. Part (a) follows from Lemma 11 and Remark 1. Part (b) follows from parts (a) and (c) of Theorem 3 and Remark 1. Part (c) follows from part (c) of Theorem 3 and parts (a) and (b) above; and this completes the proof of the corollary.

THEOREM 4. *Let $(A, \mathscr{B}_A, m)$ be a measure space, and $R \in \mathscr{E}_{III}(A)$. Let the measure space $(X, \mathscr{B}, m)$, the transformation $S \in \mathscr{D}(X)$, and the collection of transformations $\mathscr{T}$ be constructed as in Example 4. Then*

(a) *$\mathscr{T}$ does not contain any transformation $T \in \mathscr{E}_I(X)$;*

(b) *if $\mathscr{T}$ contains a transformation $T \in \mathscr{E}_{II}(X)$ then $[R]$ contains a transformation $V \in \mathscr{E}(A)$ such that V preserves a measure $\mu \sim m$ on $(A, \mathscr{B}_A)$;*

(c) *if $[R]$ contains a tranformation $V \in \mathscr{E}(A)$ which preserves a measure $\mu \sim m$ on $(A, \mathscr{B}_A)$ and such that μR is piecewise linear with respect to μ, then $\mathscr{T}$ contains a transformation $T \in \mathscr{E}_{II}(X)$.*

PROOF. Part (a) follows from Lemma 11.

To prove part (b) we let $T \in \mathscr{T}$ such that $T \in \mathscr{E}_{II}(X)$, and we let $\mu \sim m$ be an invariant measure for T on $(X, \mathscr{B})$. For $y \in A = (A,0)$ let $T_0 y = T^n y$ where $n = n(y)$ is the smallest positive integer such that $T^n y \in A$. If we denote T_0 by V on $(A, \mathscr{B}_A, m)$ then it is clear that $V \in \mathscr{E}(A)$, $V \in [R]$, and V preserves the measure $\mu \sim m$ on $(A, \mathscr{B}_A)$. This proves part (b).

To prove part (c) we let $\mu \sim m$ be an invariant measure for some ergodic transformation $V \in [R]$ on $(A, \mathscr{B}_A)$, and such that μR is piecewise linear with respect to μ. We let $A = \bigcup_{\alpha \in \Lambda} A_\alpha$(disj) where $\Lambda = \{\alpha \in \mathbf{R} \mid \mu(A_\alpha) > 0\}$ and $A_\alpha = \{y \in A \mid (d(\mu R)/d\mu)(y) = \alpha\}$ for $\alpha \in \mathbf{R}$. Since $\Delta = \Delta(\mu) =$ the group generated by Λ is countable, we enumerate the elements of Δ by $\mathbf{Z}$ and denote it by $\Delta = \{\delta_n \mid n \in \mathbf{Z}\}$ where $\delta_0 = 1$. For $x = (y, n) \in X$ where $y \in A$ and $n \in \mathbf{Z}$ we let $T(y,n) = (Ry, j)$ where $(d(\mu R)/d\mu)(y) = \delta_k$ and $\delta_n \delta_k = \delta_j \in \Delta$. It follows that $T \in \mathscr{T}$. Next we extend to $(X, \mathscr{B})$ the measure $\mu \sim m$ on $(A, \mathscr{B}_A)$, denote it by the same letter μ, and define it by $\mu[(B,n)] = \delta_n^{-1}\mu(B)$ for $B \subset A$ and $n \in \mathbf{Z}$. It follows that $\mu \sim m$ on $(X, \mathscr{B})$. Let $(B,n) \subset (A,n)$ where $B \subset A$ and $n \in \mathbf{Z}$, then $(B,n) = \bigcup_{k \in \mathbf{Z}} (B_k, n)$ where $B_k = \{y \in B \mid (d(\mu R)/d\mu)(y) = \delta_k\}$ for $k \in \mathbf{Z}$. We then have $\mu[T(B_k,n)] = \mu[(RB_k, j)]$ where $\delta_j = \delta_n\delta_k$; but $\mu[(RB_k, j)]$ $= \delta_j^{-1}\mu(RB_k) = \delta_j^{-1}\delta_k\mu(B_k) = \delta_n^{-1}\mu(B_k) = \mu[(B_k,n)]$. This shows that μ is an invariant measure for T on $(X, \mathscr{B})$. Finally, we show that $T \in \mathscr{E}(X)$. For $y \in A = (A, 0)$ we let $T_0 y = T^n y$ where $n = n(y)$ is the smallest positive integer such that $T^n y \in A$. We note from Lemma 8 that for almost all $y \in A$ such a positive integer n exists. Moreover, $T_0 \in [R]$ and $[T_0] = \{Q \in [R] \mid \mu Q = \mu\}$. Since $V \in [R]$ is ergodic and preserves the measure $\mu \sim m$ on $(A, \mathscr{B}_A)$ it follows that $T_0 \in \mathscr{E}(A)$. Similarly, for $k \in \mathbf{Z}$ and $(y,k) \in (A,k)$ if we let $T_k(y,k) = T^n(y,k)$ where $n = n[(y,k)] > 0$ is the smallest positive integer such that $T^n(y,k) \in (A,k)$, then

$$T_k \in \mathscr{E}[(A,k)] \text{ for } k \in \mathbf{Z}. \tag{12}$$

It is then easy to see from (12) and Lemma 8 that $T \in \mathscr{E}(X)$, and this completes the proof of the theorem.

COROLLARY. *Let $(A, \mathscr{B}_A, m)$ be a measure space and let $R \in \mathscr{E}_{III}(A)$. Suppose there exists an ergodic transformation $V \in [R]$ which preserves a measure $\mu \sim m$ on $(A, \mathscr{B}_A)$ and such that μR is piecewise linear with respect to μ. Let the measure $\nu \sim m$ on $(A, \mathscr{B}_A)$ be piecewise linear with respect to μ. Then νR is piecewise linear with respect to ν, $\Delta(\mu) \subset \Delta(\nu)$, and there exists an ergodic transformation $U \in [R]$ which preserves ν if and only if $\Delta(\mu) = \Delta(\nu)$.*

PROOF. For almost all $y \in A$ we have

$$\frac{d(\nu R)}{d\nu}(y) = \frac{d\nu}{d\mu}(Ry) \cdot \frac{d(\mu R)}{d\mu}(y) \cdot \frac{d\mu}{d\nu}(y);$$

and since the measures ν, μR, and μ are piecewise linear with respect to the measures μ, μ, and ν, respectively, it follows that νR is piecewise linear with respect to ν.

Since ν is piecewise linear with respect to μ, there exists a positive real number $\alpha \in \boldsymbol{R}$ such that $B_\alpha = \{y \in A \mid (d\nu/d\mu)(y) = \alpha^{-1}\}$ has positive measure. From the ergodicity of the transformation $V \in [R]$ it follows that for almost all $y \in A$ there exists a positive integer $n \in N$ such that $V^n y \in B_\alpha$. We also have

$$(13) \qquad \frac{d(\nu V^n)}{d\nu}(y) = \frac{d\nu}{d\mu}(V^n y) \cdot \frac{d(\mu V^n)}{d\mu}(y) \cdot \frac{d\mu}{d\nu}(y).$$

Since $V^n y \in B_\alpha$, V preserves the measure $\mu \sim m$ on $(A, \mathscr{B}_A)$, and $(d(\nu V^n)/d\nu)(y) \in \Delta(\nu)$ we conclude from (13) that there exists a positive real number $\alpha \in \boldsymbol{R}$ such that

$$(14) \qquad \alpha \frac{d\nu}{d\mu}(y) \in \Delta(\nu) \quad \text{for all } y \in A.$$

For $\beta \in \Delta(\mu)$, then from (14) it follows that

$$(15) \qquad \beta = \frac{d(\mu R^n)}{d\mu}(y) = \left[\frac{d\nu}{d\mu}(R^n y)\right]^{-1} \cdot \frac{d(\nu R^n)}{d\nu}(y) \cdot \frac{d\nu}{d\mu}(y) \in \Delta(\nu)$$

for some $y \in A$ and $n \in N$, and this shows that $\Delta(\mu) \subset \Delta(\nu)$.

Next we suppose that $\Delta(\mu) = \Delta(\nu)$. We enumerate $\Delta(\mu)$ by $\mathbf{Z}$, and construct the measure space $(X, \mathscr{B}, \mu)$, the collection of transformations $\mathscr{T}$, the transformation $S \in \mathscr{D}(X)$, and the ergodic transformation $T \in \mathscr{T}$ which preserves the measure μ as described in the proof of part (c) of Theorem 4. For $x \in X$ where $x = (y, n) \in (A, n)$ we let $P(y, n) = (y, j)$ where $\delta_j^{-1}\delta_k = \delta_j^{-1}$, $\alpha(d\nu/d\mu)(y) = \delta_k \in \Delta(\mu)$, and $\alpha > 0$ is the constant defined in (14). It follows then that $P \in [S]$, the transformation $T' = P^{-1}TP \in \mathscr{T}$, $T' \in \mathscr{E}(X)$, and the transformation T' preserves the measure $\mu P \sim \mu$ on $(X, \mathscr{B})$. For the S-section $A = (A, 0)$ we have $A = \bigcup_{j \in \mathbf{Z}} C_j(\text{disj})$ where $C_j = \{(y,0) \in (A,0) \mid P(y,0) = (y,j)\}$ for $j \in \mathbf{Z}$. We also have $(\mu P)(C_j) = \mu(PC_j) = \delta_j^{-1}\mu(C_j)$ where $\delta_j^{-1} = \alpha(d\nu/d\mu)(y)$ for $j \in \mathbf{Z}$ and $y \in A$. It follows that $\mu P = \alpha\nu$ on $(A, \mathscr{B}_A)$. Therefore, the induced transformation U on the set A by T' defined via the first return time to A under T' is ergodic and preserves the measure $\alpha\nu \sim \mu$ on $(A, \mathscr{B}_A)$, and since $\alpha > 0$ is a constant, U also preserves the measure $\nu \sim \mu$ on $(A, \mathscr{B}_A)$.

Conversely, suppose that the measure ν is piecewise linear with respect to μ and that there exists an ergodic transformation $U \in [R]$ such that U preserves $\nu \sim \mu$ on $(A, \mathscr{B}_A)$. We already have that $\Delta(\mu) \subset \Delta(\nu)$. Interchanging the roles of μ and ν in (15) we conclude that $\Delta(\nu) \subset \Delta(\mu)$ also, and this completes the proof of the corollary.

REMARK 2. From the work of W. Krieger [8], [9] on the weak equivalence of transformations the following result seems to hold. Let $(A, \mathscr{B}_A, m)$ be a measure space and let $R \in \mathscr{E}_{III}(A)$. Suppose that there exists an ergodic $V \in [R]$ such that V preserves $\nu \sim m$ on $(A, \mathscr{B}_A)$. Then, according to the terminology used in [8], there exists a measure $\mu \sim m$ on $(A, \mathscr{B}_A)$ such that R contains the measure μ; in other words, the measure $\mu R \sim m$ on $(A, \mathscr{B}_A)$ is piecewise linear with respect to μ and $\mathscr{H}(\mu) = \{Q \in [R] \,|\, \mu Q = \mu\}$ is ergodic. It is desirable to obtain a direct proof of this result without making use of the heavy machinery used in [8] and [9]. Finally, combining this fact with parts (b) and (c) of Theorem 4 and making use of Lemma 7 and the corollary to Theorem 1 it is possible to obtain the following sharper result replacing parts (b) and (c) of Theorem 4:

(b′) $\mathscr{T}$ contains a transformation $T \in \mathscr{E}_{II}(X)$ if and only if $[R]$ contains a transformation $V \in \mathscr{E}_{II}(A)$.

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